19.11.2024

Paradoxical sets

and the

Axiom of Choice

Montreal Descriptive Dynamics and Combinatorics Seminar

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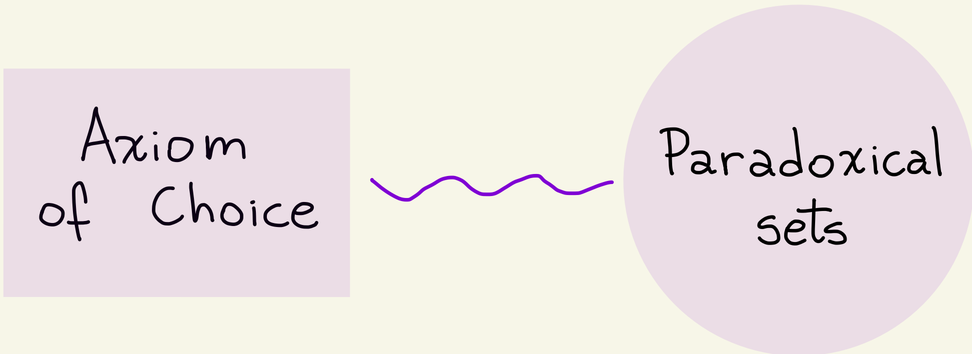
University of Münster

Context



Axiom of Choice

Axiom of Choice

$$\Longleftrightarrow$$

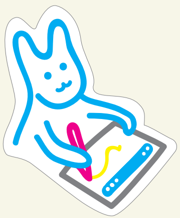
Zorn's lemma

$$\Longleftrightarrow$$

Well-ordering principle

$$\Downarrow$$

well-ordering on the reals



Paradoxical sets

- $P \subseteq \mathbb{R}^n$ $\quad n = 1, 2, 3.$

P satisfies some counterintuitive property.

The existence of such a P involves Axiom of Choice

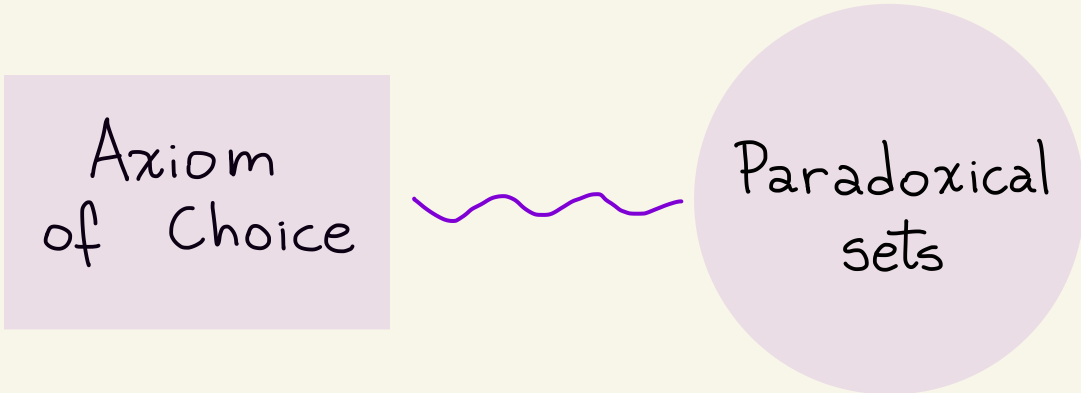
$$\Downarrow$$

well-ord. of $\mathbb{R}$

$$\updownarrow$$

transfinite induction on the reals

Context



Context

Axiom of Choice

well-order of the reals

...

?

Paradoxical sets

Vitali set

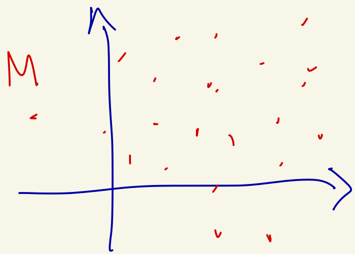
Hamel basis

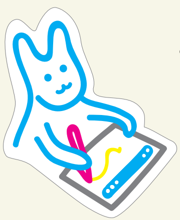
Mazurkiewicz set

PUC

Partitions of $\mathbb{R}^3$ Unit Circles

Maz set





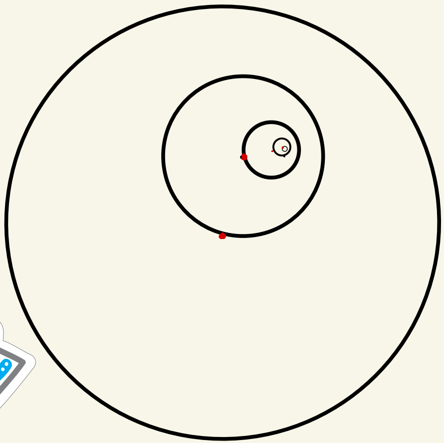

TOMMY

Partitions in circles

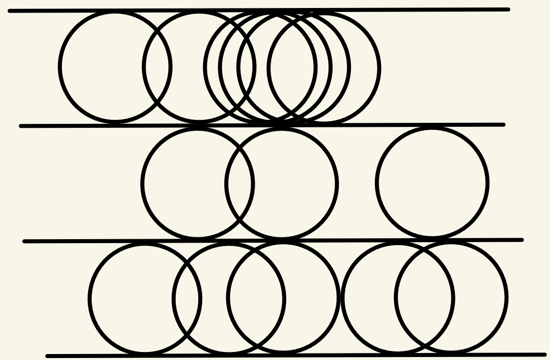
PUC := partition of $\mathbb{R}^3$ in unit circles

Questions
(i) Why $\mathbb{R}^3$?
(ii) Why partition? 1- covering
(iii) Why unit circles?

(i)



(ii) 2- covering



Partitions in circles

Theorem (ZF) (Szulkin)
$\mathbb{R}^3$ can be partitioned in circles.

Theorem (ZFC) (Conway-Croft / Kharazishvili)
$\mathbb{R}^3$ can be partitioned in unit circles.

Partitions in circles

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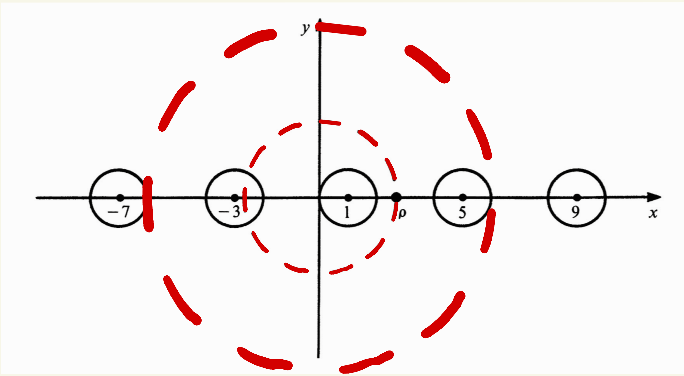
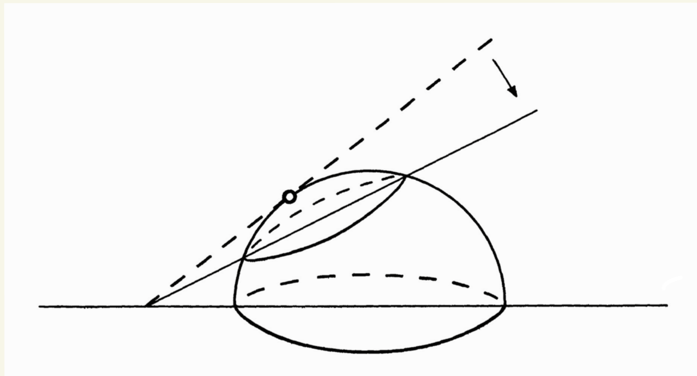
$\mathbb{R}^3$ can be partitioned in circles.



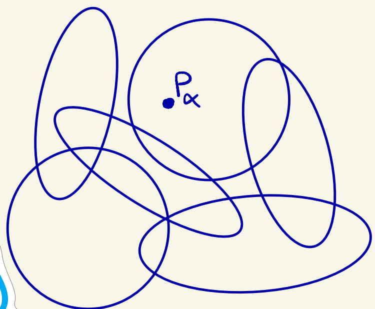


Partitions in unit circles

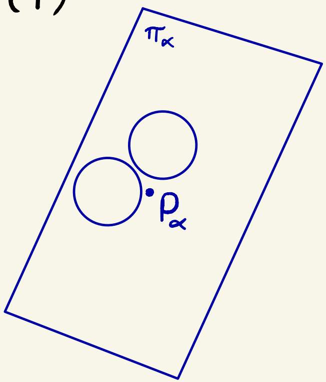
Theorem (ZFC) (Conway-Croft / Kharazishvili)
$\mathbb{R}^3$ can be partitioned in unit circles.

$$\mathbb{R}^3 = \{p_\alpha\}_{\alpha < \mathfrak{c}}$$

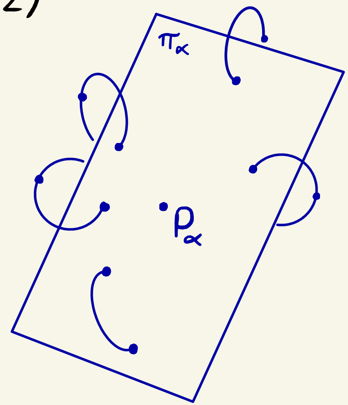
(0) step α



(1)



(2)

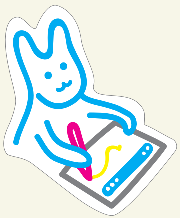


Partitions in unit circles

Observation: The proof shows that any partial PUC of cardinality $<\mathfrak{c}$ can be extended to a (complete) PUC.

Question: Can any partial PUC be extended to a PUC?

- No! There could be not enough space!



Context

mathoverflow

Concerning proofs from the axiom of choice that $\mathbb{R}^3$ admits surprising geometrical decompositions: Can we prove there is no Borel decomposition?

(…) *So the general situation is that the axiom of choice constructions are both easy and flexible and not entirely ungeometrical.*

Partitions in circles

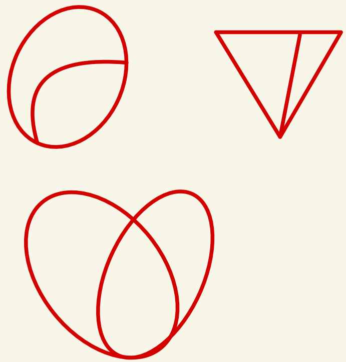
PUC := partition of $\mathbb{R}^3$ in unit circles

- If analytic, then Borel
- If $V=L$, then there is a coanalytic PUC

Set theory

- Model of ZF(C) = "universe" → There are several

  C ↓ Axiom of Choice

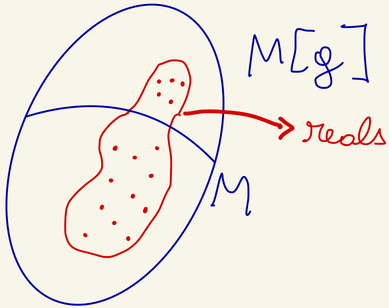


How does constructing models look like?

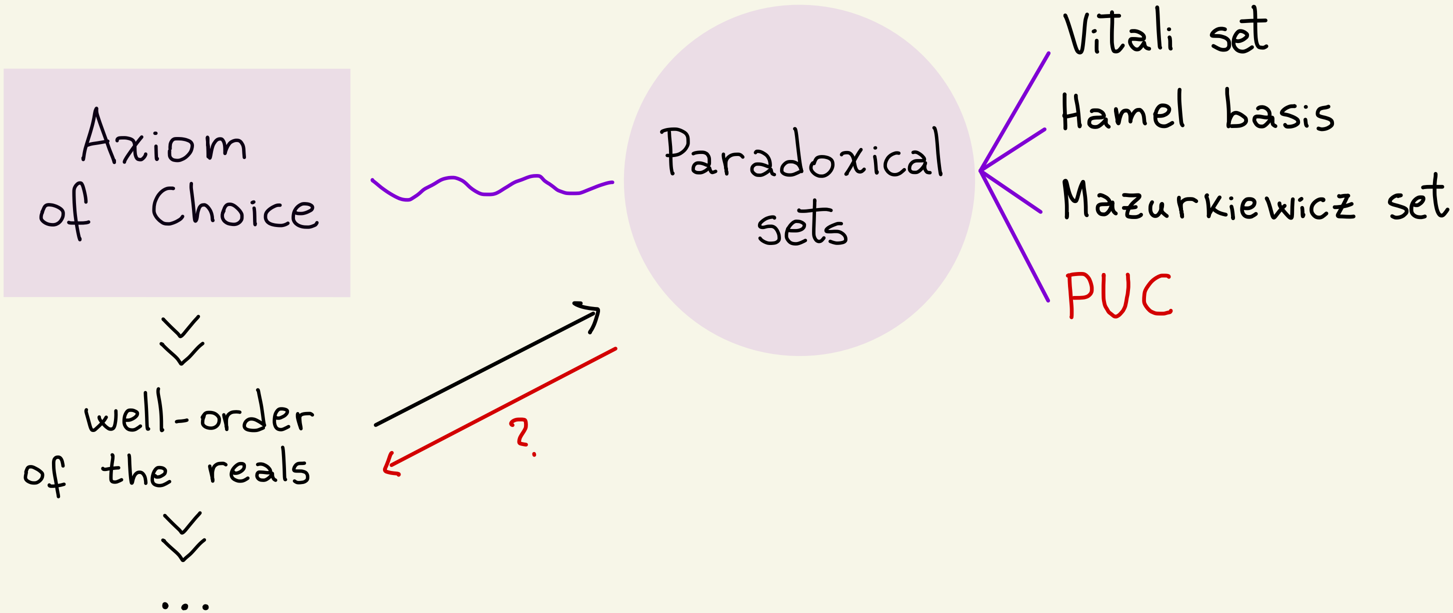
Set theory

How does constructing models look like?

$$(M,\ \mathbb{P},\ g \subseteq \mathbb{P}) \longrightarrow \boxed{\text{Forcing}} \rightsquigarrow M[g]$$



Context


Axiom of Choice
Paradoxical sets
Vitali set
Hamel basis
Mazurkiewicz set
PUC
well-order of the reals
?
...

The result(s)

Theorem

There is a model of

$$ZF + \exists \text{ PUC} + \neg \text{ well-order on } \mathbb{R}$$

The model(s)

1. Cohen-Halpern-Lévy model:

$$H := \mathrm{HOD}^{L[g]}_{A \cup \{A\}}$$

where g is $\mathbb{C}(\omega)$-generic over L, and

$A = \{c_n : n < \omega\}$ is the set of Cohen reals added by g.

2.

$$W = L(\mathbb{R}, b)^{L[\tilde{g}, h]}$$

where $\tilde{g}$ is $\mathbb{C}(\omega_1)$-generic over L,

h is $\mathbb{P}$-generic over $L[\tilde{g}]$, and

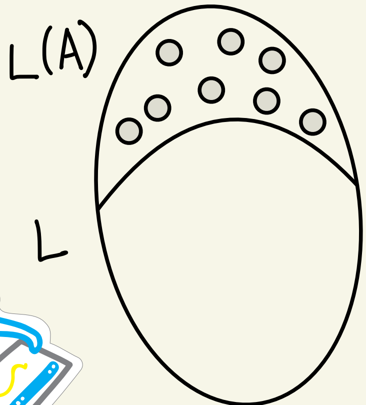
$b = \bigcup h$ is the PUC added by h.

Cohen-Halpern-Lévy model

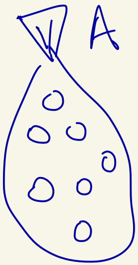
1. Cohen - Halpern - Lévy model:

$$H := L(A)$$

where g is $\mathbb{C}(\omega)$-generic over L, and
A is the set of Cohen reals added by g.



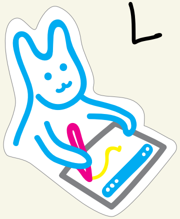
● ≙ Cohen real



A infinite $\subseteq \mathbb{R}$

○ + Z $\in \mathbb{R}$

$r \to$ a real in $L(A)$
only needs finitely
many elements of A



Cohen-Halpern-Lévy model

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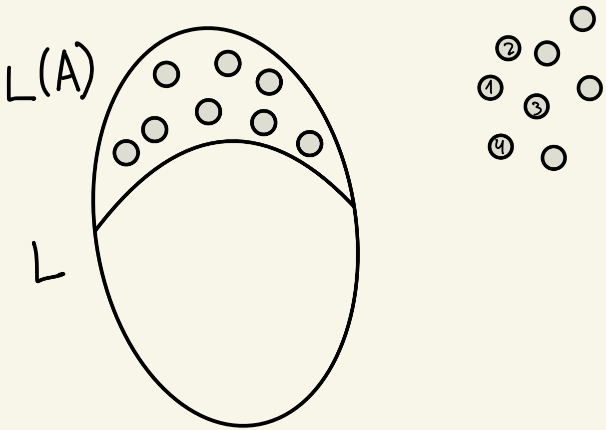
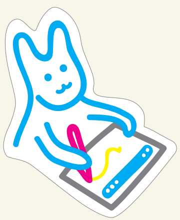




Cohen-Halpern-Lévy model

1. Cohen-Halpern-Lévy model:

$$\boxed{H := L(A)}$$

Facts about H

(i) There is no well-ordering of the reals.

(ii) There is no countable subset of $A \subseteq \mathbb{R}$

(iii) $\mathbb{R} \cap H = \bigcup_{a \in [A]^{<\omega}} (\mathbb{R} \cap L[a])$

Recent results

Models of $ZF + \neg C + \exists P$

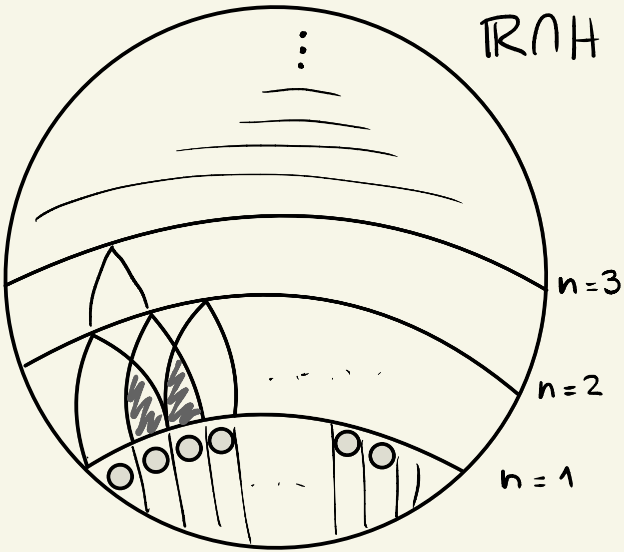
- First Cohen model $L(A)$:

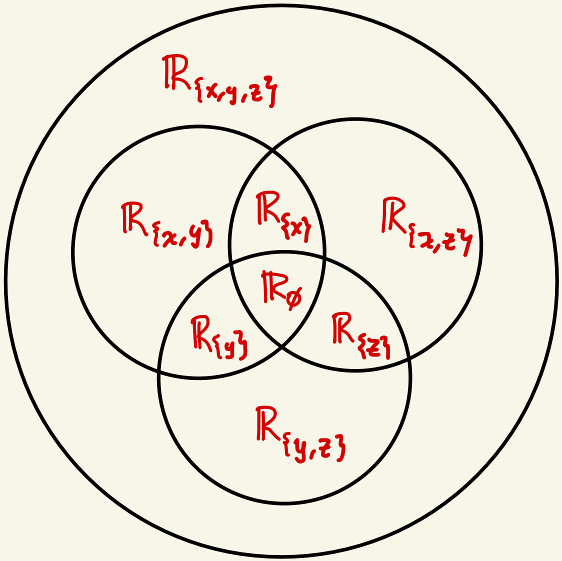
  $ZF + \neg AC_\omega + \ldots$ + Hamel basis [BSWY]
  + Mazurkiewicz set [BS]
  + Partition $\mathbb{R}^3$ in unit circles [F]

Construction of a PUC in H

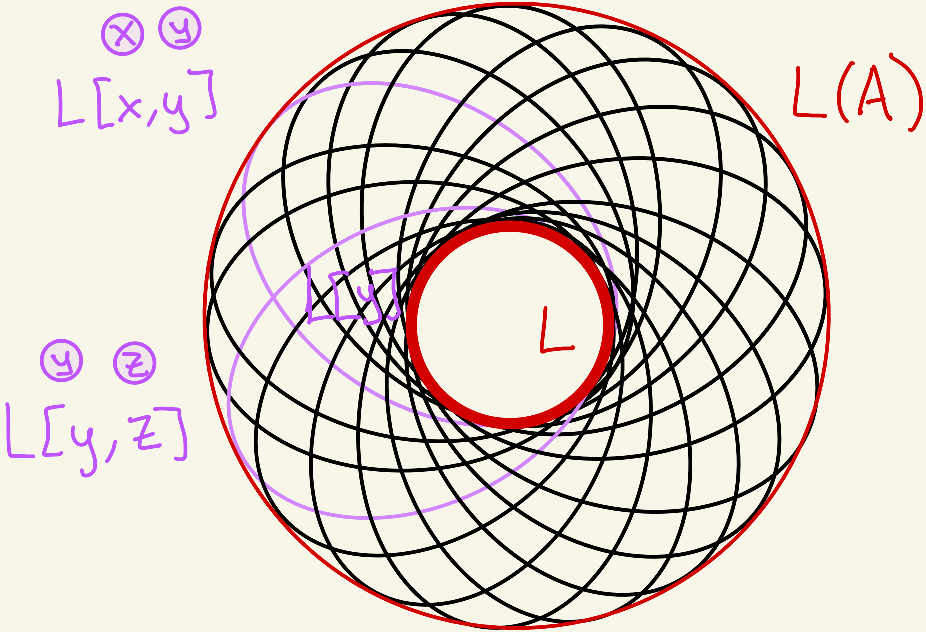
Q: How do we get a PUC in H?

$$\mathbb{R}\cap H = \bigcup_{a\in[A]^{<\omega}} (\mathbb{R}\cap L[a]) = \bigcup_{n<\omega} \bigcup_{\substack{|a|=n \\ a\subseteq A}} \mathbb{R}_a$$



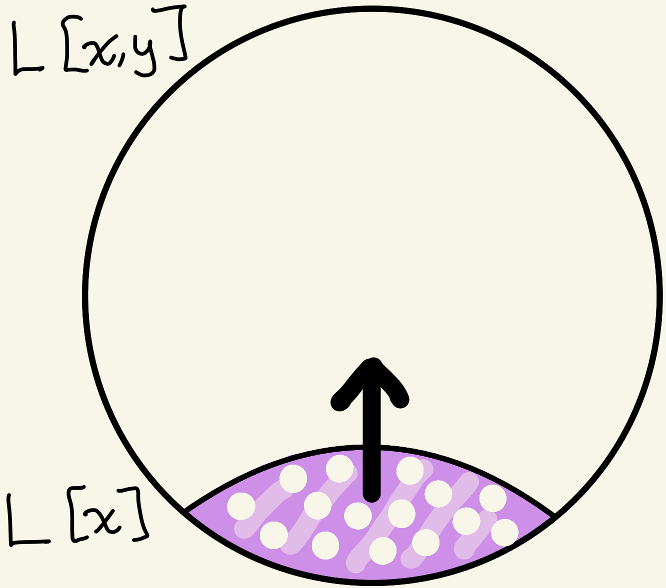


Construction of a PUC in H

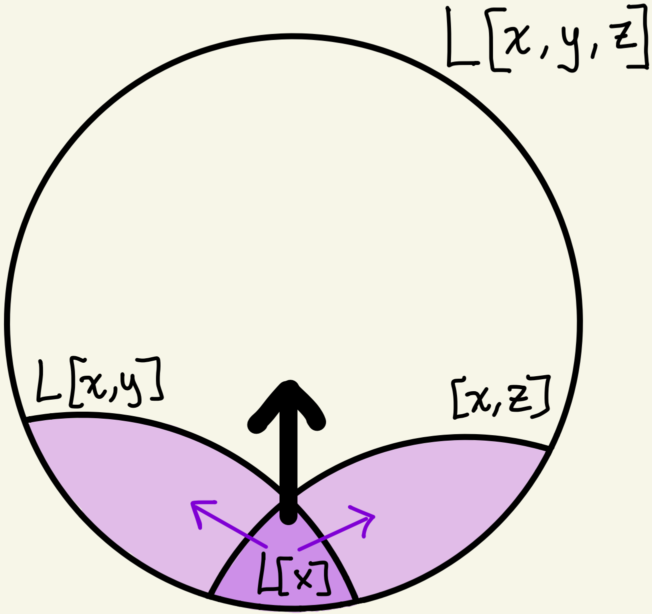


Strategy: use AC in each inner model to construct PUC's, and be careful to glue them well →

Obstacles to glue the PUCs



Extendability



Amalgamation